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ABSTRACT

The question of how Piaget selects mathematical concepts for his theory is addressed as a problem which may have contributed to a misinterpretation of the relevance of conservation studies for mathematics education. A new form of pseudo-conservation is discussed in which nonconservers of length are able to pick out strips of equal length from an irregular array. How Piaget thinks of mathematical concepts is discussed as being related to his general epistemological stance which is compatible with a creative, heuristic view of mathematics and not with a process or skill view. (MP)

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Mathematical Foundations of Forty Years of Research on Conservation in Geneva*

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During the past four decades, in dozens of publications, Piaget and his colleagues in Geneva have sought to obtain a clear picture of the cognitive structures and mechanisms that lead children autonomously to conservation beliefs in discrete and continuous quantities. Their unrelenting effort, which includes the continual invention of new tasks and the exploration of mathematical structures (groups, lattices, and now categories) seems inspired by two fundamental epistemological problems: (1) How is it possible for a biological species to develop mathematics? And (2) How is it possible for abstract mathematical formulations to contribute to man's understanding of the physical world? (Cf, Piaget, 1967c). His stance on the first problem is indicated by Piaget (1965) in this statement:

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> The problem of logico-mathematical knowledge comes midway between that of knowing how knowledge is acquired by heredity (a subject to which one was bound to revert in order to understand the biological nature of knowledge at the higher levels), and how it is acquired by learning--a process which is of necessity acquired and not hereditary. Logico-mathematical knowledge belongs to neither of these categories and yet it is a necessary part of the second one. (English translation, 1971, pp. 303-306).

This stance, that mathematical knowledge is neither hereditary nor acquired, in turn sharpens the second problem. For if it is primarily a construction by the individual, its applicability must mean it is constructed out of experience manipulating objects in a practical way.

Efforts to clarify the specific uses Piaget has made of mathematical and logical structures in his theory have been necessary because he does

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not follow standard mathematical forms of argument. His uses, indeed, have proved somewhat resistant to standard analysis, which fact suggests that they may be unusual uses. Writers who have worked on this problem include Piaget, Grize, Apostel, Mays, Papert, Beth, Parsons, Granger, Witz, Flavell, Easley, Lunzer, Ennis, Wermus, Wittmann, Steiner, Szoke, Beilin, Steffe, Rotman, and doubtless many others. There are also references in print to conversations involving other mathematicians and philosophers about the difficulties of conceptualizing Piaget's uses of mathematics and logic. As an interested participant in this effort, I feel that a fundamental problem sometimes overlooked is how Piaget's epistemological approach relates to his choice of mathematical concepts. I shall address this foundational question, but I shall not attempt to put this rather incoherent literature in order--a larger task than I feel competent to undertake. However, by attempting to provide perspective on the epistemological purposes mathematical structures serve in Piaget's program, I hope to interest more of the mathematics education research community in the deeper epistemological problems on which Piaget is working. By focussing on conservation studies, I hope to throw some light on the problems that have puzzled educators in relating them to mathematics education, including the questionable idea that conservation is a prerequisite to school mathematics.

There can be no doubt of the importance Piaget attaches to mathematics in his program. Besides the many references to the epistemological problems mentioned above, he argues that mathematics should serve genetic epistemology much as it does theoretical physics--that, instead of using statistical models as many psychologists do, he prefers to use mathematical and logical

structures which bear a closer relation to the subject matter of human thought and knowledge. But this function is still ambiguous. For example: Do statistical models represent the object of inquiry or the inquiring subject? In what sense are groups, Piaget's own groupings, lattices, and categories involved in the development of conservation? This is the principal question I want to try to answer.

Piaget's strong mathematical interests continue up to the present time. For example, in his address to the Jean Piaget Society in 1975, Piaget (1976) said that he was making a major change in his theory: transformations are no longer seen as a sufficient basis for the development of intelligence. Inspired by their role in mathematical category theory, he now gives correspondences and morphisms an equally central theoretical position. He reported that this idea led to empirical studies which showed that demonstrating to young children the correspondence between positive and negative aspects of a transformation (adding on and taking away, which he calls the commutability of a transformation) accelerates the development of conservation by two to three years. Also, in another review of recent developments in his work (1977), Piaget mentions the contribution of Wermus (1972) and Wittmann (1973, 1975) showing that groupings can be considered as categories. Another mathematical idea, which plays a key role in this revised theory of conservation (reported in Inhelder, et al., 1975) is the concept of envelopings, a psychological version of the mathematical concept of envelope (See Easley, 1978). The usual omission of any reference to Piaget's mathematical interests in the current educational literature is a flaw which may contribute to a serious mis-interpretation of the educational significance of conservation,

for we see that mathematical structures play a key role in current research in Geneva.

I have alluded to a mis-application of conservation in mathematics education. Let me be specific here. I refer to the claim made by many writers (e.g., Lovell and Shayer, 1978; Steffe, 1976; Copeland, 1970, 1974) that, without conservation of a particular quantity, schoolwork involving that quantity reduces to making meaningless, artificial connections. I should like to point out some things that Piaget says that undercut this claim and then show that his epistemological research program is aimed to defeat the perspective that makes this claim attractive. It is necessary to clear the air on this point, because the connection between mathematics and Piaget's work cannot be seen clearly in the perspective from which this claim arises.

On the surface, it sounds plausible that a child who no longer believes that two identical rulers have the same length when one of them has been moved along its length (into a staggered position) might encounter trouble when using a ruler that is moved along its length in order to measure something. On a deeper level, the claim seems to be that, if discrete or continuous quantities are believed to change when transformations of a collection or an object are being carried out, then the continuity assumed in making calculations with such quantities is missing. Procedures of calculation are seen as depending on quantitative properties of objects or of assemblies of objects that change in synchrony with arithmetic operations and not otherwise. Without such a linkage between arithmetic and physical transformations, arithmetic operations are said to lack physical meaning and thus, unless one postulates abstract referents, reduce to the manipulation of strings of marks,

"tokens," according to conventional pattern rules. Some writers, strangely, appear to look to Piaget's work for support of the use of manipulative aids to provide concrete referents for so-called "abstract symbols." The symbols themselves are concrete marks on paper or chalkboard. Their usual referents are abstract. Piaget's theory emphasizes reflective abstraction from one's own physical actions in the formation of logico-mathematical mental structures. Representing perceptual properties of concrete objects or collections of objects, such as their length or their cardinal number, is more nearly what Piaget calls figurative knowledge, or possibly physical abstraction. What he calls concrete operations is the application of internalized logico-mathematical transformations or correspondences, one-at-a-time, to objects or their internal representations. Formal operations involve group-like systems of transformations and correspondences. Both may involve the manipulation of physical objects or not, as circumstances permit.

That Piaget does not share the more common perspective can be inferred from his response (1968) to Mehler and Bever (1967) in which they claimed to find conservation of number in two and one-half to three-year-old children. (See also Beilin, 1968, for further discussion of this issue.) Piaget (1968) writes as follows:

> The natural tendency of young children, evidently, is to conserve as long as they are not confronted by facts which they do not expect, and whose inexplicability leads them to change their opinion.

He then connects these experiments on number, (which Mehler, Bever, and Epstein, 1968, in their reply to Piaget, 1968, concede had been incorrectly described as conservation tasks) with experiments on continuous quantity in which

a screen hid the transformation of pouring into a taller, narrower glass and children were asked to predict the results.

> Before they have seen what actually happens these children expect that there will still be "the same amount to drink," that the quantity of liquid will be conserved, and the level of the liquid will be conserved as well (in fact, the level of the liquid constitutes the measure of its quantity from an ordinal point of view). (Piaget, 1968)

Thus, Piaget seems to believe such children do connect the level of liquid with their idea of quantity in a way that makes sense to them, and it is only their surprise that the level is not as they expect in certain situations that makes them abandon their natural belief in conservation. He goes on,

> In these cases we speak about pseudoconservation, since these subjects conserve too much, and therefore incorrectly. Only after having seen that the level goes up in the narrow glass will these subjects deny conservation of quantity and opine that the quantity has increased because the level has changed; yet, to repeat the point, this is not simply due to a perceptual strategy, but to the fact that, in the absence of other means of measurement, an ordinal evaluation by level necessarily leads to this conclusion.

(For other discussions of pseudoconservation, see Leahy, 1974, 1977; Piaget and Inhelder, 1966; Piaget, 1967a). That Piaget thinks of nonconservers as approaching the world in a meaningful, constructive way, and not in a meaningless, rote way, is made perfectly clear by his summary:

> In general, children expect conservation, but since they cannot know beforehand what will be conserved and what will not be conserved, they have to construct new means of quantification in every new sector of experience. The inadequacy of the means of quantification explains nonconservation, and it is worth noting that nonconservation therefore indicates an effort to analyze and to dissociate variables; very young children and severely mentally retarded subjects pay no attention to these variables, whereas the older, normal children pass through a stage of nonconservation as they reorganize relations which they cannot yet grasp in full. (Piaget, 1968)

The principled, nonrote, use of counting for quantifying collections of objects by 2-5 year olds is well described by Gelman and Gallistell (1978), but they ignore Piaget's distinction between psuedo-conservation and operational conservation. I should like to report another, striking example of pseudo-conservation, or at least an expectation of conservation in the absence of transformations, discovered by one of my collaborators in a project on concepts of number and numeral in primary-grade children. It supports a broader view of cognitive functioning than one gets by focussing exclusively on Piaget's groupings as operational structures. Bernadine Stake examined kindergarten children who did not conserve length in the task in which two elongated objects, identical in appearance (pencils, Cuisenaire rods, rulers, etc.) were first placed side-by-side with their ends together and then one of them was moved along its length into a staggered position. She then presented each child with an irregular array of strips of tagboard and asked him/her to pick out pairs of strips which were of the same length. Many of these nonconservers were able immediately to pick out two identical strips which were lying next to each other in the same staggered arrangement of the other objects they had previously declared unequal. That it was a different task for these children when the objects were static and when they were moved was made quite clear by posing a series of further questions about the same pair of strips the subjects had just identified as equal in length. For example, to escape the influence of this response, they were asked whether they were still equal when held behind the experimenter's back, when placed on the table in the form of a "T," then in a 7-shape, then placed together end-to-end, and finally, one was moved so they were

in the same staggered position in which the child had originally recognized their equality. Now, in the last position, after having given a series of yes or no answers to the other tasks, these children always said the two strips were different in length.

I should like to try to explain this apparent instability of the conservation judgment, but first I should like to add that my collaborators and I have repeatedly asked nonconservers of length over a transformation what they could do with the ruler, which they had just declared changed its length when it was moved, and a pencil. None had any difficulty in saying they could measure the pencil and in identifying the nearest number of inches to its length. In this operation, it was clear that the previous movement of the ruler past another ruler had no relevance.

In fact, this provides the clue to my explanation, which draws on Piaget's own work on movement (1946). Piaget recognizes this application himself in the same paper quoted above (1968).

> Research on the concept of length (of paths, for example) has shown that, at the level of development in question, "longer" means "going further," not because of a dominance of perception, or because of verbal or semantic confusion, but because the first quantifications that become possible before the synthesis of number (a synthesis of order and inclusion) and that of measure (a synthesis of partition and displacement) are based on an order of the points of arrival.

It is very clear, in our clinical interviews on length, that children who are nonconservers where the transformation is involved often maintain a sense of direction, as though the two objects are racing and one is overtaking the other or falling behind it. The object that is moved by the experimenter is not always said to be the longer one. That depends on

the direction in which the racing is imagined to take place. The "longest" is the one that comes out ahead, given the direction of the race.

The inspiration for Piaget's research on movement and speed he once (1964) attributed to a suggestion Einstein made to him, that it would be interesting to know which developed first in children, concepts of distance and time or the concept of speed. There seems to be a connection between this question and Einstein's special theory of relativity, in which distance and time intervals are dependent on the relative velocities of the frames of reference in which they are measured. If children develop conservation of speed before they develop conservation of distance and time, it would suggest that Einstein's theory, in which the speed of light is absolute and distance and time are relative to the observer's frame of reference, is more natural than classical physics in which speed is defined in terms of absolute distance and time, taken as primitive terms. Indeed Piaget's work seems to confirm this hypothesis, but that takes us too far afield.* What is relevant here is that what we might call the "racing schema" is a very early scheme that is quite strong in most four to seven-year-old children. (They race eating their cereal, doing their homework, and have a similar competitive feeling about who is taller, older, etc., as well as in the more classical sense of running, pushing toy cars, etc.)

The relevance of the racing schema for the conservation of length, number, amount to drink, etc. is that this early, comparative concept of

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* Bohm (1965) discusses Piaget's work in an appendix to his text on The Special Theory of Relativity and comes to the conclusion that Einstein's theory is psychologically more natural than classical mechanics.

quantitative relations is used in a wide variety of situations. It is easy for children to see the clinical interview tasks on number, quantity, length, etc. as involving a "race" or competition in which the questions are assimilated, the judgment of who is getting ahead or who is falling behind, regardless of what quantitative words (same, more than, less than, etc.) and regardless of other information such as counting, measuring, and one-to-one correspondence. Piaget models what we call the "race schema" in terms of functions. He writes,

> Before the child becomes capable of reversible operations, his thinking proceeds by "functions" in the modern sense of "mappings" (one-way mappings) or of "ordered couples." Psychologically, functions are the expression of action schemes, and every action (particularly an action whereby a certain distance is covered) is a series of ordered movements which will come to an end (at the point of arrival). This concept of "function" explains the dominance of ordinal considerations that underlie quantitative evaluations of and by length between 3 to 4, and 6 to 7 years of age. (Piaget, 1968).

This makes it clear that nonconservation is far from a meaningless, artificial way of interpreting quantitative relations. Indeed, in asking our kindergarten subjects to do arithmetic problems, adding and subtracting one digit numbers, we found children who are nonconservers on number and length who can handle these problems well in a variety of different presentations and materials and others who are conservers who cannot. Clinically, we cannot trace any connection between conservation and elementary arithmetic. The fact that pseudoconservers have two schemes for interpretation that give conflicting answers in the interview task does not mean they will be in conflict in school or everyday situations.

Turning now to Piaget's way of using modern mathematical ideas to explain these universal concepts of quantity, we have a problem already in the last quote. Does Piaget think of a function as a set of ordered pairs, no two of which have the same first component? Piaget says he's using the modern sense of one-way mappings or ordered couples, placing quotes around the latter term. We have a great deal of evidence elsewhere that he wishes to avoid the notions of set, class, and attribute, because of their Platonistic connotations (See Easley, 1978). So, I would infer, that instead of defining a function as a set of ordered pairs, he thinks of it as one or more causally "ordered" or connected movements. Moving this string, moves this weight on the end of it (Piaget, et al., 1968). So the whole Boole-Frege approach is avoided. This includes the two sets normally referred to as the domain and co-domain of the function, as well as their Cartesian product, and the particular sub-set of their Cartesian product we use to define the function. This does not mean that Piaget recommends avoiding all use of sets in teaching children, or in interpreting what children are thinking. Rather it means that he does not want sets to be reified as Platonistic ideals or even for them to have the status of foundational concepts to which other concepts may be reduced. Piaget and Szeminska make this concern clear in numerous passages in their book on the origins of number in children (1941). For example, they write in response to Russell's definition of cardinal number as a class of classes in 1-1 correspondence,

> . . .if we assert that any element in F can correspond to any element in F_1 we have the right to conclude that F and F_1 correspond numerically term for term, and that this

> correspondence defines the number six. This number is not a 'class of classes,' but the result of a new operation brought in from outside, which is not contained in the logic of classes as such. In fact, this 'quantifying' correspondence is only achieved by disregarding all the attributes in question, i.e., by disregarding the classes (Eng. Trans., 1952, p. 183).

And, in the introduction to the same book, they wrote:

> Nevertheless, the connections established in the field of experimental psychology needed to be verified in the field of logistics, and we proceeded to attempt this verification.
>
> In studying the literature on the subject we were surprised to find to what extent the usual point of view was 'realist" (i.e., Platonist) rather than 'operational,' with the exception of the interesting work of A. Reymond. This fact accounts for the connections, many of them artificially established by Russell, which forcibly separated logistic investigation from psychological analysis, whereas each should be a support for the other in the same way as mathematics and experimental physics. (Eng. Trans., 1952, p. ix)

Piaget wants a logistic analysis that is oriented toward the observations he makes of children and, presumably, toward explaining the natural origins of mathematical operations rather than reifying them as timeless patterns on a substrate of ideal objects, sets.

Another purpose often associated with mathematical structures is that of defining a closed system. It is clear that certain groups of transformations can "define" a new conceptual entity like a space or a geometry--and in this sense the whole generated by that structure is much greater than the sum (in a merely collective sense) of the parts of the individual transformations. In the long run, Piaget could look for similar integrations of parts into structured wholes heuristically to generate such new conceptions in the child. However, the group structures of "displacements" and of additions and subtractions of integers are invoked by Piaget without

concern whether children conceive or even are capable of conceiving these systems as closed or not. Piaget frequently defines groups and other structures without reference to closure. What function do groups serve in his theory then? Clearly he sees such structures as providing stability and organization in the continuing process of cognitive reorganization. Such stability depends on a relative degree of integration, not on absolute closure. Certainly an infinite set of integers is not a part of the young child's concept of number. However, moving around only among the familiar numbers, recognizing what adding and subtracting mean relative to each other has a structure that needs representing. The group concept is what comes closest for me. But it does have one piece of excess baggage, closure. Piaget utilized Felix Klein's Erlangen program in which transformation groups (isometries) define geometries, and worked out the theory of the INRC group of transformations on logical formulas, and category theory was hailed by Bourbaki as the new Erlangen program for algebra.

One recognizes implicit Platonism in the axiom systems of Grize and Apostel because they use meta-mathematical language, referring to sets of objects and mappings between them. Grize recognized some of these problems of domains that his (1960) axiomatization had not solved and in 1964 proposed using MacLane's notion of a category, in which one could separate the operations as a complete algebraic structure from the concrete objects to which the structure applies (personal correspondence). Piaget (1967b) mentioned categories as the direction to which it would be promising to turn, and Papert (1967) developed the idea somewhat further. It is clear that it would solve the closure problem, for categories make no such requirement.

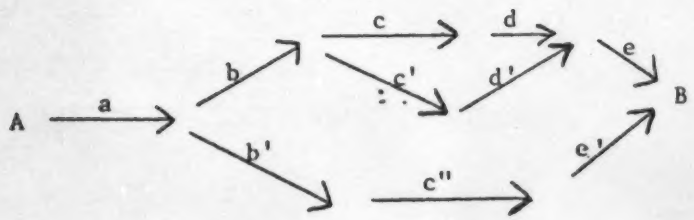
If we are going to ever make sense of Piaget's mathematical theorizing, I believe it will come from abandoning the dominant Boole-Frege concepts. Some encouragement for this comes from MacLane who says,

> Since a category consists of arrows, our subject could also be described as learning how to live without elements, using arrows instead . . . which covers . . . the means to prove all the diagram lemmas without ever chasing an element around a diagram." (1971, p. v)

MacLane refers to a category of graphs of arrows with limited composition, but including identity operations, as a "metacategory,"(1971, p. 1). Higgins contrasts the two main perspectives of category theory thus:

> The bulk of category theory aims at providing general theorems with applications in concrete categories (or in similar ones whose objects may be more complicated mathematical structures and whose morphisms may be, for example families of mappings). . .
>
> From a slightly different point of view one may also regard a category as an algebraic structure in its own right, on the same footing as a group or a semigroup. The "elements" of this algebra are the morphisms, and their composition is an associative partial binary operation; the objects act as labels for the morphisms in order to determine the domain of the operation. Functors are now to be thought of as structure-preserving maps, i.e., algebra homomorphisms. (1970, p. 2.)

Indeed, this less concrete version of category theory seems ready-made for Piaget's epistemological preferences. As Steiner (1973, 1974) and Whittmann (1975) show, many arithmetical systems (e.g., all combinations of natural numbers making 8, Papy's minicomputer, etc.) are groupings in their generalized sense and representable as categories. It would seem that Piaget's description of a function, quoted above, would be describable as a category and that, as reversibility is added, such structures move closer to groupoids (categories "in which every arrow is invertible," (MacLane, 1971, p. 20). A series of ordered movements that take one from



A to B, to which Piaget gives the name of function, may be simple and direct a, b, c, d, e, or may include alternative routes c', d', or b', c", e'. By adding this commutativity, the graph becomes a little more interesting. Identity arrows are needed to make it a category, but Piaget never seems to have difficulty interpreting identity psychologically. To take this route, however, and be drawn into psychologically accurate detail, may be to give up some of the conceptual power of transformation groups. The power of category theory also is not its flexibility but its abstractness. So Piaget (1976) may be understandably trying to create a blend of transformations and correspondences--to have some of the best of both worlds, so to speak, but moving cautiously so as not to lose what he has already gained.

Closed systems like the addition group of integers clearly represent a bigger step forward than most children are ready to take at the point at which they begin to add. We don't expect children at this stage to understand that integers are indefinitely numerous. The problem has been to describe finite arithmetical systems that are open, yet well-defined. A category such as the one described by Steiner (1973) for combinations of Cuisenaire rods is a structure children can be taught before they have any idea of how the whole system of counting numbers is constructed. Also,

at this stage one really has no need of the general ability to form a length seriation, although it is easy enough for most children to make a Cuisenaire "staircase." That suffices, but in objects where the differences are not so clearly visible and the whole envelope of the staircase is not evident, seriation by successive and systematic pair-wise orderings is really difficult. Certainly, there is no indication that conservation of number over spatial transformations might be related to these categories.

Given Piaget's epistemological goal to explain how mathematics itself is psychologically possible and useful, these applications of category theory are only suggestive. In fact, Piaget cannot, without circularity appeal to the conceptual power that mathematicians recognize in categories and transformation groups. This is, he cannot both use mathematical structures as models of the development of ideas in children and use the facts about children's development, so represented, to explain how mathematical ideas of the same sort arise psychologically. I believe that I read Piaget correctly in seeing him as taking the second option, but in this paper I am following the other route as a heuristic strategy for trying to see connections which are otherwise obscure. Having looked down the other road myself (Easley, 1977; Driver and Easley, 1978), I am also choosing the easier path. Perhaps, once I get clear on the connection from mathematics to the new experiments with children I can try it again in the preferred direction. I should like to propose a way in which the category idea can be applied to one of the new experiments on conservation.

In one of the experiments claimed most successful in advancing the age of conservation and showing how conservation develops spontaneously

(Inhelder et al., 1975, Piaget, 1976), bits are broken off, one at a time, from a ball of plasticine and then placed on opposite sides of the ball to lengthen it into a sausage shape. With each bit, the child is asked whether breaking it off makes the ball smaller than the other ball that is left untouched, and whether adding it on again will make it the same again. Eventually, by taking off bits around the middle and building up two ends the ball is transformed into a somewhat irregular "sausage." Some non-conserving children with this training are reported to transfer their acquired conservation to the case where the sausage is produced by rolling the ball on the table.

What is claimed in the new theory, is that when a system of correspondences is grasped between the original positions of bits of plasticine in the ball and the final positions of the same bits in the sausage, then conservation is accepted. This principle that what is taken away compensates precisely for what is added is called commutability, as a generalization of the arithmetic principle of commutativity. When this schema of correspondences is developed, I would say the racing schema of judging one ball "the winner" if it is longer or thicker than the other is no longer evoked by the task.

Such a system of correspondences, shown in Figure 1, can be seen to be a morphism, a mapping that preserves the whole ball of plasticine. Clearly, not every order of composition is possible, because bits that are removed first cannot be put at the ends of the sausage, which haven't been built up yet. Bits removed from the ball's inner layers can't be added on first, unless, of course, one abandons the one-bit-at-a-time procedure and

takes all the bits off first and then adds them on in some new order. If you did that, however, presumably the correspondence (commutability) the child is in the process of building up would be lost, and conservation would not be attained. There are, of course, many other possible morphisms than the one diagramed in Figure 1, and thinking about alternative morphisms leads in the direction of the arrow graph in Figure 2. With identity arrows and commutativity, it is a category. What is useful is its general form, omitting particular bits of plasticine. The generalization to a category from one morphism is equally possible from all others. Thus the generalization of invariance of substance over all such transformations of shapes is like the generalization from a morphism to a category which forgets the particular elements, the particular concrete objects (bits) involved in each arrow, and just looks at the structure of the process.

Thus, the important consideration is not to generate a complete set of morphisms with all possible permutations. It is really uninteresting how many there might be with eight pieces because, of course, a different number of pieces of plasticine might be moved in the very next try. What is interesting is to see beyond the particular morphism that was carried out. The fact that a category permits certain compositions and not others and represents alternate sequences of actions gives it the right sort of tightness or looseness to represent this generalization. Representing continuous transformations is another category.

Inhelder, et al., (1975) claim that conservation of number and continuous quantities are structurally isomorphic. However, the transformations they carry out on rows of discrete but continuous counters do not seem to evoke

the racing schema as much as one transformation they do not mention. That is shown in Figure 3. Here, by moving a counter from one end to the other end of a single row, one row can be seen to be gaining or to be falling behind in the "race" depending, of course, on which way the race is seen as going. If the training procedure is carried out using this transformation, there is a natural confusion in using the "racing schema," because removing a counter from one end of a row is inevitably seen as shortening it, whichever end it is taken away from. Then adding it on to the opposite end may be seen as making that row longer or making it shorter depending on how the race is now seen as going. Reversing the race a few times may be a way of dispelling it. But what is the structure of conservation or compensation here?

In summary, let me say that Piaget's epistemological concerns still control his research on conservation and rather naturally lead him to categories. (For other treatments of these particular epistemological problems, Beth and Piaget, 1961, and Rotman, 1977, Ch. 5). The only clear application of his work then to mathematics education would be an epistemological one, in which the whole conception of mathematics as abstract objects and conventional rules and procedures would be replaced by an action-oriented view of creative thought. But many educators see "creative mathematics" as a contradiction in terms--at least at the elementary school level. As long as this is the case, there is no particular relevance in conservation testing, and certainly no teacher should ever withhold instruction in arithmetic, measurement or other customary curricular ideas because the child or children concerned have not demonstrated their belief in the conservation of the quantities concerned.

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Abstract

Paiget's work on conservation and the mathematical ideas he brings to it seem inspired by his quest to understand the biological development of mathematics and the applicability of mathematics to real world problems. How he selects mathematical concepts for his theory and, in particular, the reasons for his recent interest in category theory have not been made clear, a problem which may contribute to a misinterpretation of the relevance of conservation studies for mathematics education. The widespread opinion that conservation is needed for attaching physical meaning to arithmetic is challenged by Piaget's own seldom quoted statements of the reasons for nonconservation and especially for psuedo-conservation and by some clinical evidence from research in progress.

A new form of pseudo-conservation is reported, in which nonconservers of length are able to pick out strips of equal length from an irregular array even though they are in the same relative position (staggered) in which they are declared to be unequal when one of them has just been moved from the aligned position. This is seen as related to Piaget's work on movement, to his studies of functions as precursers of operations, and to what is here called the "racing" schema--a general childhood organizational pattern. Nonconservation in these cases is seen as a natural form of quantification which overrides the expectation of conservation.

How Piaget thinks of mathematical concepts is related to his general epistemological stance, which is anti-Platonistic and action based. This view makes categories much more natural for him than the Boole-Frege set concepts which are standard in mathematical representations. To avoid

circularity in presentation, Piaget either has to settle on using mathematical structures as models of children's ideas or use children's ideas to explain the origins of mathematical ideas. While he seems more and more to take the second option, this paper takes the first perspective in order to deal with the question of mathematical representation. One recent experiment which is claimed to advance conservation by several years (Inhelder, et al., 1975) is shown to be representable by a morphism and a category. This is not a unique representation but a heuristic on explaining how such a training procedure might work, by helping the child avoid the "racing" schema. Piaget's mathematical epistemology is compatible with a creative, heuristic view of mathematics, but not with a process or skill view such as dominates education today.

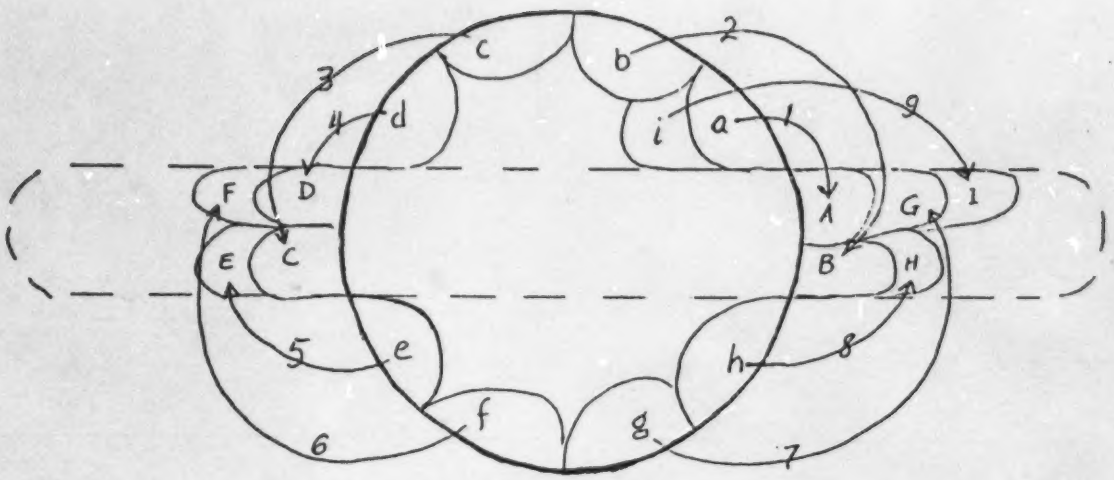

Figure 1. Beginning a transformation of a plasticine ball into a "sausage" one bit at a time--a morphism.

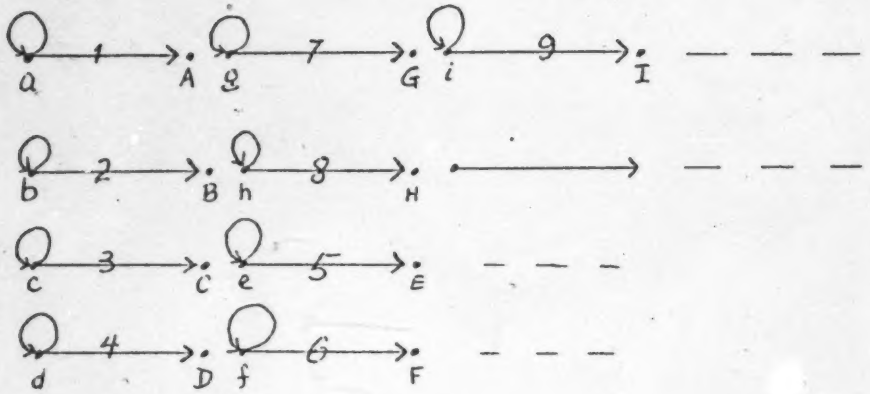

Figure 2a. Beginning construction of a system of arrows generalizing the structure of the morphism depicted in Fig.1, by representing essential linkages. The identity arrows at the base of each numbered arrow represent the possibility of returning each piece of plasticine to its original position --a possibility of which non-conserving children need to be reminded.

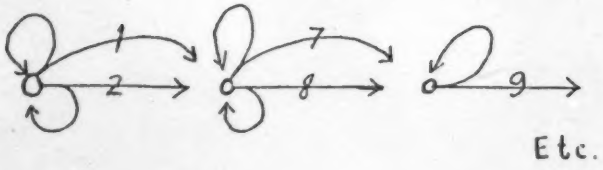

Figure 2b. A further stage in the construction begun in Fig. 2a. The emerging structure generalizes over the many morphisms possible by (1) combining the bases of the arrows that are optional at a given time and (2) forgetting the particular points of origin (a, b, c, ...) of each preserving only those linkages of arrows required by the layers of bits in the ball and in the "sausage". The structure emerging here is a category.

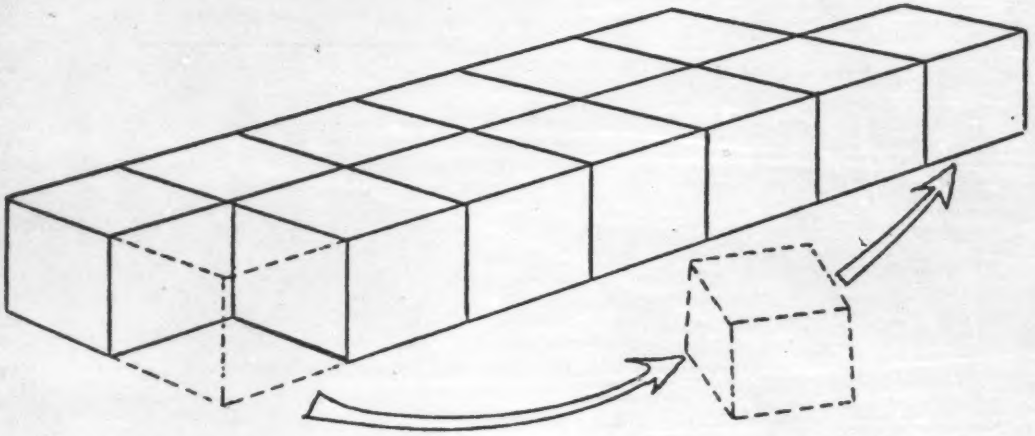

Figure 3. Simple apparatus for demonstrating commutability.